it is quite legitimate to assume all labour homogeneous. There will be other applications still to notice as we go on.[1]

For the present, we shall only use this principle to assure ourselves that the classification of the effects of price on demand into income effects and substitution effects, and the law that the substitution effect, at least, always tends to increase demand when price falls, are valid, however the consumer is spending his income.

5. In all our discussions so far, we have been concerned with the behaviour of a single individual. But economics is not, in the end, much interested in the behaviour of single individuals. Its concern is with the behaviour of groups. A study of individual demand is only a means to the study of market demand. Fortunately, with our present methods we can make the transition very easily.

Market demand has almost exactly the same properties as individual demand. This can be seen at once if we reflect that it is the actual change in the amount demanded (brought about by a small change in price) which we can divide into two parts, due respectively to the income effect and the substitution effect. The change in the demand of a group is the sum of changes in individual demands; it is therefore also divisible into two parts, one corresponding to the sum of the individual income effects, the other to the sum of the individual substitution effects. Similar propositions to those which held about the individual effects hold about the group effects.

(1) Since all the individual substitution effects go in favour of increased consumption of the commodity whose price has fallen, the group substitution effect must do so also.

(2) Individual income effects are not quite reliable in direction; therefore group income effects cannot be quite reliable either. A good may, of course, be inferior for some members of a group, and not be inferior for the group as a whole; the negative income effects of this section being offset by positive income effects from the rest of the group.

(3) The group income effect will usually be negligible if the

[1] Beyond this, it does not seem necessary to worry about the definition of a 'commodity'. What collections of things we regard as composing a commodity must be allowed to vary with the problem in hand.

group as a whole spends a small proportion of its total income upon the commodity in question.

6. We are therefore in a position to sum up about the law of demand. The demand curve for a commodity must slope downwards, more being consumed when the price falls, in all cases when the commodity is not an inferior good. Even if it is an inferior good, so that the income effect is negative, the demand curve will still behave in an orthodox manner so long as the proportion of income spent upon the commodity is small, so that the income effect is small. Even if neither of these conditions is satisfied, so that the commodity is an inferior good which plays an important part in the budgets of its consumers, it still does not necessarily follow that a fall in price will diminish the amount demanded. For even a large negative income effect may be outweighed by a large substitution effect.

It is apparent what very stringent conditions need to be fulfilled before there can be any exception to the law of demand. Consumers are only likely to spend a large proportion of their incomes upon what is for them an inferior good if their standard of living is very low. The famous Giffen case, quoted by Marshall,[1] exactly fits these requirements. At a low level of income, consumers may satisfy the greater part of their need for food by one staple foodstuff (bread in the Giffen case), which will be replaced by a more varied diet if income rises. If the price of this staple falls, they have a quite considerable surplus available for expenditure, and they may spend this surplus upon more interesting foods, which then take the place of the staple, and reduce the demand for it. In such a case as this, the negative income effect may be strong enough to outweigh the substitution effect. But it is evident how rare such cases must be.

Thus, as we might expect, the simple law of demand—the downward slope of the demand curve—turns out to be almost infallible in its working. Exceptions to it are rare and unimportant. It is not in this direction that our present technique has anything new to offer.

7. But as soon as we pass beyond this standard case, we do begin to get some effective clarification.

[1] *Principles*, p. 132.

So far we have assumed the consumer's income to be fixed in terms of money. What happens if this is not so, if he comes to the market not only as a buyer but also as a seller? Suppose he comes with a fixed stock of some commodity X, of which he is prepared to hold back some for his own consumption, if price-conditions are favourable to that course of action.

It is clear that so long as the price of X remains fixed, our previous arguments are unaffected. We may suppose, if we like, that he exchanges his whole stock into money at the fixed price, when he will find himself in exactly the same position as our consumer whose income was fixed in terms of money. He can then buy back some of his X if he wants to.

But what happens if the price of X varies? The substitution effect will be the same as before. A fall in the price of X will encourage substitution of X for other goods; this must favour increased demand for X, that is to say, diminished supply. But the income effect will not be the same as before. A fall in the price of X will make a *seller* of X worse off; this will diminish his demand (increase his supply) unless X is for him an inferior good.

The significant difference between the position of the seller and that of the buyer thus comes out at once. In the case of the buyer income effect and substitution effect work in the same direction—save in the exceptional case of inferior goods. In the case of the seller, they only work in the same direction in that exceptional case. Ordinarily they work in opposite directions.

The position is made more awkward by the fact that sellers' income effects can much more rarely be neglected. Sellers usually derive large parts of their incomes from some particular thing which they sell. We shall therefore expect to find many cases in which the income effect is just as powerful as the substitution effect, or is dominant. We must conclude that a fall in the price of X may either diminish its supply or increase it.

The practical importance of such a supply curve is no doubt most evident in the case of the factors of production. Thus a fall in wages may sometimes make the wage-earner work less hard, sometimes harder; for, on the one hand, reduced piece-rates make the effort needed for a marginal unit of output seem less worth while, or would do so, if income were unchanged; but on the other, his income is reduced, and the urge to work harder in order to

make up for the loss in income may counterbalance the first tendency.[1]

Such a supply curve will appear, however, whenever there is a possibility of reservation demand; that is to say, whenever the seller would prefer, other things being equal, to give up less, rather than more. The supply of agricultural products from not too

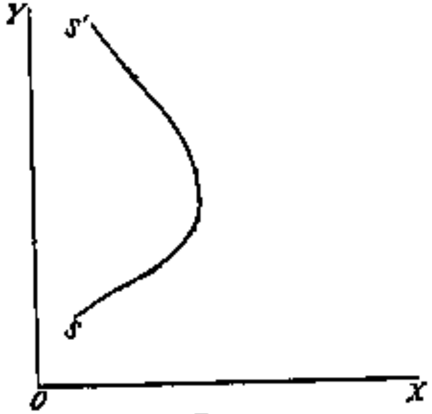

FIG. 9.

specialized farms is thus another good example. Any such supply curve, drawn on a price-quantity diagram, is likely to turn back on itself at some point. We cannot be at all confident that it will be upward-sloping (Fig. 9).

That there existed this asymmetry between supply and demand has long been familiar; it should perhaps be reckoned as one of the discoveries of Walras.[2] But so long as the reason for the asymmetry was not made clear, it was rather too easy to forget its existence. To have cleared up this matter may be regarded as the first-fruits of our new technique. It is itself a good thing to have cleared up, and, we shall find as we go on, it opens the way to some very convenient analytical methods.

[1] Robbins, 'Elasticity of Demand for Income in Terms of Effort' (*Economica*, 1930, p. 123).

[2] Walras, *Éléments d'économie politique pure* (first published 1874), leçons 5–7.

would not fall regularly; so that the occurrence of utterly disastrous slumps would, in these circumstances, be rather probable. I do not think one could count upon the long survival of anything like a capitalist system, using that term to mean a system of free enterprise, including free lending and borrowing.[1]

We began our study of dynamic economics by rejecting the concept of a stationary state as an analytical tool. We rejected it then, because it seemed to be no more than a special case, which offered no facility for generalization. We have come in the end to doubt whether it is even conceivable as a special case; to suspect that the system of economic relations we have been studying is nothing else but the form of a progressive economy.

[1] The reasons which have led many people to suppose that this sort of danger is likely to be actual in the twentieth century are, of course, the practical cessation of geographical discovery and the approaching fall in population. These are weighty reasons; yet the trend of innovation in the future is, by its very nature, so difficult to forecast that we cannot deduce imminent peril from these things alone. Nevertheless, one cannot repress the thought that perhaps the whole Industrial Revolution of the last two hundred years has been nothing else but a vast secular boom, largely induced by the unparalleled rise in population. If this is so, it would help to explain why, as the wisest hold, it has been such a disappointing episode in human history.

MATHEMATICAL APPENDIX

1. The purpose of this Appendix is not merely the transcription of the argument of the text into mathematical symbols; I see little advantage to be got from doing that. When the verbal (or geometrical) argument is conclusive, it gains nothing from being put in another form. What can be gained, however, is the assurance that our argument is completely general; that what has been proved in the text for two, or three, or four, commodities, is true for n commodities. In this Appendix I shall concentrate upon the proof of that generality.

I shall follow the same order of subjects as in the text of the book, and shall mark off the sections of the Appendix according to the chapters of the book to which they refer. I must begin, however, by giving some discussion of a purely mathematical proposition, which is fundamental to what follows. Its relevance will appear almost at once.

2. *A fundamental mathematical proposition.* (1) The general homogeneous function of the second degree in three variables

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

can also be written in the form

$$a\left(x+\frac{h}{a}y+\frac{g}{a}z\right)^2+\frac{ab-h^2}{a}\left(y-\frac{gh-af}{ab-h^2}z\right)^2+$$
$$+\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}(z)^2.$$

Since the variables appear only within the brackets, and each bracket is squared, it appears at once that the original expression is positive for all real values of the variables if the coefficients of all brackets are positive, negative if the coefficients are all negative. These coefficients are ratios of the determinants

$$a,\quad \begin{vmatrix} a & h \\ h & b \end{vmatrix},\quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Thus the original expression is definitely positive if all three determinants are positive, definitely negative if the first and third are negative and the second positive.

(2) A similar proposition can be established for any number of variables.[1] The general quadratic form

$$a_{11}x_1^2+a_{22}x_2^2+\ldots+a_{nn}x_n^2+2a_{12}x_1x_2+2a_{13}x_1x_3+\ldots+2a_{23}x_2x_3+\ldots$$

will be positive for all real values of the x's if the determinants

$$a_{11},\quad \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix},\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix},\ldots,\quad \begin{vmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{12} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{1n} & a_{2n} & . & . & a_{nn} \end{vmatrix}$$

are all positive, negative if they are alternatively negative and positive.

(3) If it is required to find the conditions that the above quadratic form should be definitely positive or negative, not for all values of the variables, but for those values only which satisfy the linear relation

$$b_1x_1+b_2x_2+\ldots+b_nx_n = 0,$$

we can proceed by eliminating one of the variables, say x_1. The quadratic form then becomes

$$c_{22}x_2^2+c_{33}x_3^2+\ldots+c_{nn}x_n^2+2c_{23}x_2x_3+\ldots,$$

where $$c_{rs} = a_{rs}-\frac{1}{b_1}(a_{1r}b_s+a_{1s}b_r)+\frac{1}{b_1^2}b_rb_sa_{11}.$$

The required conditions can then be written in the same form as that given in (2) above, with c's in place of the a's; but they can be simplified if we multiply every determinant by the necessarily negative quantity $-b_1^2$. For example,

$$-b_1^2\begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} = \begin{vmatrix} 0 & b_1 & 0 & 0 \\ b_1 & a_{11} & 0 & 0 \\ b_2 & a_{12} & c_{22} & c_{23} \\ b_3 & a_{13} & c_{32} & c_{33} \end{vmatrix} = \begin{vmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33} \end{vmatrix},$$

adding appropriate multiples of the first two columns to each of the remaining columns.

Thus the conditions for the quadratic form being definitely positive

[1] Cf. Burnside and Panton, *Theory of Equations*, vol. ii, pp. 181–2.

subject to a linear condition are that the determinants

$$\begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix},\quad \begin{vmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33} \end{vmatrix},\ldots,\quad \begin{vmatrix} 0 & b_1 & b_2 & . & . & b_n \\ b_1 & a_{11} & a_{12} & . & . & a_{1n} \\ b_2 & a_{12} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_n & a_{1n} & a_{2n} & . & . & a_{nn} \end{vmatrix}$$

should be all *negative* (since the negative factor $-b_1^2$ will change all signs); the conditions for its being definitely negative are that the determinants should be alternatively positive and negative.

This is all we need as a purely mathematical foundation; let us now turn to the economics.

Appendix to Chapter I

3. *Equilibrium of the consumer*. We begin by considering an individual, who has a given sum of money M available for expenditure (call it provisionally his 'income') and has opportunities for spending it upon n different commodities. The prices of these n commodities are given to him as determined on the market. Call them $p_1, p_2, p_3,\ldots, p_n$. Call $x_1, x_2, x_3,\ldots, x_n$ the amounts of the respective commodities which he buys.

Then, provided he spends all his income, we must have

$$M = \sum_{r=1}^{r=n} p_r x_r. \qquad (3.1)$$

Assume for the moment that his wants are expressed by a given utility function $u(x_1, x_2, x_3,\ldots, x_n)$. The amounts bought will be determined by the condition that u is a maximum, subject to the condition (3.1). They can be worked out by introducing a Lagrange multiplier μ, and maximizing:

$$u+\mu\Big(M-\sum_{r=1}^{r=n} p_r x_r\Big).$$

The conditions for consumer equilibrium are therefore that

$$u_r = \mu p_r \quad (r = 1, 2, 3,\ldots, n), \qquad (3.2)$$

where u_r is written for $\partial u/\partial x_r$, the *marginal utility* of x_r. The equation thus expresses the equality between the marginal utility of x_r and the price of x_r multiplied by μ (which is accordingly identified as Marshall's *marginal utility of money*).

When μ is eliminated between the equations (3.2), they reduce to

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \dots = \frac{u_{n-1}}{p_{n-1}} = \frac{u_n}{p_n}. \tag{3.3}$$

These $n-1$ equations, together with the equation (3.1), provide n equations to determine the n quantities $x_1, x_2, \dots, x_n$.

4. *Stability conditions.* In order that u should be a true maximum it is necessary to have not only $du = 0$ (as above) but also $d^2u < 0$. Expanding these expressions, and writing u_{rs} for the second partial derivative, as u_r for the first, we have

$$du = \sum_{r=1}^{r=n} u_r\,dx_r,$$

$$d^2u = \sum_{r=1}^{r=n}\sum_{s=1}^{s=n} u_{rs}\,dx_r\,dx_s.$$

This latter expression is a quadratic form of the same character as that discussed in § 2 above (since $u_{sr} = u_{rs}$); consequently the conditions for $d^2u < 0$ for all values of $dx_1, dx_2, \dots, dx_n$, such that $du = 0$, are that the determinants

$$\begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{vmatrix}, \quad \begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & u_{11} & u_{12} & u_{13} \\ u_2 & u_{12} & u_{22} & u_{23} \\ u_3 & u_{13} & u_{23} & u_{33} \end{vmatrix}, \dots, \quad \begin{vmatrix} 0 & u_1 & u_2 & \cdot & \cdot & u_n \\ u_1 & u_{11} & u_{12} & \cdot & \cdot & u_{1n} \\ u_2 & u_{12} & u_{22} & \cdot & \cdot & u_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & u_{1n} & u_{2n} & \cdot & \cdot & u_{nn} \end{vmatrix} \tag{4.1}$$

should be alternatively positive and negative.

These determinants will play an exceedingly important part in our subsequent analysis. I shall write the last of them U; and the co-factors of u_r, u_s, u_{rr}, u_{rs}, in U, I shall denote by U_r, U_s, U_{rr}, U_{rs}. Since the n goods can be taken up in any order, it follows directly from (4.1) that U_{rr}/U is necessarily negative.

5. *The ordinal character of utility.* The equilibrium conditions and the stability conditions for an individual consumer have been written out assuming the existence of a particular utility function u. This is, indeed, the most convenient way of writing them; but it is important to observe that they do not depend upon the existence of any unique utility function. For suppose the utility function u to be replaced by any arbitrary function of itself $\phi(u)$. Then it can be shown that, provided only the function $\phi(u)$ increases when u increases—that is to say, provided $\phi'(u)$ is positive—the equilibrium conditions and the stability conditions will be entirely unaffected by the change in the utility function.

Since $\frac{\partial}{\partial x_r}\phi(u) = \phi'(u).x_r$, the equilibrium conditions (3.3) will be unchanged. The equal ratios are simply multiplied by a common factor $\phi'(u)$, which cancels out. (Even if they are written in the form (3.2), they are still unchanged, provided that μ is replaced by $\phi'(u).\mu$. Since μ is arbitrary, it is legitimate to make this alteration.)

Since $\frac{\partial^2}{\partial x_r \partial x_s}\phi(u) = \phi'(u).u_{rs} + \phi''(u)u_r u_s$, the stability determinants reduce down similarly. The first determinant becomes

$$\begin{vmatrix} 0 & \phi'(u)u_1 & \phi'(u)u_2 \\ \phi'(u)u_1 & \phi'(u)u_{11}+\phi''(u)u_1^2 & \phi'(u)u_{12}+\phi''(u)u_1u_2 \\ \phi'(u)u_2 & \phi'(u)u_{12}+\phi''(u)u_1u_2 & \phi'(u)u_{22}+\phi''(u)u_2^2 \end{vmatrix} = \{\phi'(u)\}^3 \begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{vmatrix}$$

and the same reduction can be performed for every determinant in the series. The rth determinant in the series has $(r+2)$ rows and columns; it will therefore have to be multiplied by the factor $\{\phi'(u)\}^{r+2}$. Since $\phi'(u)$ is assumed positive, none of the determinants have their signs changed by the introduction of such a factor; and since it is the signs of the determinants which govern stability, the conditions may be considered to be unaltered by the substitution of $\phi(u)$ for u.

Thus, if we decide (as I think we should) to start, not from a given utility function, but from a given scale of preferences, all we have to do is to confine our attention to those properties of the utility function which are invariant against the substitution of $\phi(u)$ for u. The original equilibrium conditions and the original stability conditions have been shown to be invariant in this way. The remainder of our theory of value will be worked out using invariant properties only, though I shall generally leave it to the reader to check the invariance for himself.

Appendix to Chapters II and III

6. *The effect on demand of an increase in income.* Let us revert to the equilibrium equations (3.1) and (3.2), writing them in the form

$$\left.\begin{aligned} p_1x_1+p_2x_2+\dots\ +p_nx_n &= M \\ -\mu p_1+u_1 &= 0 \\ -\mu p_2+u_2 &= 0 \\ \cdot\quad \cdot\quad \cdot\quad &\cdot \\ -\mu p_n+u_n &= 0 \end{aligned}\right\}. \tag{6.1}$$

Differentiating partially with respect to M,

$$\left.\begin{aligned} p_1\frac{\partial x_1}{\partial M}+p_2\frac{\partial x_2}{\partial M}+\dots+p_n\frac{\partial x_n}{\partial M}&=1\\ -p_1\frac{\partial\mu}{\partial M}+u_{11}\frac{\partial x_1}{\partial M}+u_{12}\frac{\partial x_2}{\partial M}+\dots+u_{1n}\frac{\partial x_n}{\partial M}&=0\\ -p_2\frac{\partial\mu}{\partial M}+u_{21}\frac{\partial x_1}{\partial M}+u_{22}\frac{\partial x_2}{\partial M}+\dots+u_{2n}\frac{\partial x_n}{\partial M}&=0\\ \dots\dots\dots\dots\dots\dots&\\ -p_n\frac{\partial\mu}{\partial M}+u_{n1}\frac{\partial x_1}{\partial M}+u_{n2}\frac{\partial x_2}{\partial M}+\dots+u_{nn}\frac{\partial x_n}{\partial M}&=0 \end{aligned}\right\}. \tag{6.2}$$

Solving,

$$\frac{\partial x_r}{\partial M}\begin{vmatrix} 0 & p_1 & p_2 & \dots & p_n\\ p_1 & u_{11} & u_{12} & \dots & u_{1n}\\ p_2 & u_{12} & u_{22} & \dots & u_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ p_n & u_{1n} & u_{2n} & \dots & u_{nn}\end{vmatrix} = \begin{vmatrix} 0 & p_1 & \dots & p_{r-1} & 1 & p_{r+1} & \dots & p_n\\ p_1 & u_{11} & \dots & u_{1,r-1} & 0 & u_{1,r+1} & \dots & u_{1n}\\ p_2 & u_{12} & \dots & u_{2,r-1} & 0 & u_{2,r+1} & \dots & u_{2n}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ p_n & u_{1n} & \dots & u_{r-1,n} & 0 & u_{r+1,n} & \dots & u_{nn}\end{vmatrix}.$$

Since (6.1) $p_s = u_s/\mu$, this can be written

$$\frac{\partial x_r}{\partial M}=\frac{\mu U_r}{U}. \tag{6.3}$$

Nothing is known about the sign of U_r; consequently $\partial x_r/\partial M$ may be either positive or negative. (See above, Chapter II, pp. 27–9.)

7. *The effect of a change in price with constant income.* Now suppose p_r to vary, other prices (and M) remaining unchanged. We have from (6.1)

$$\left.\begin{aligned} p_1\frac{\partial x_1}{\partial p_r}+p_2\frac{\partial x_2}{\partial p_r}+\dots+p_n\frac{\partial x_n}{\partial p_r}&=-x_r\\ -p_1\frac{\partial\mu}{\partial p_r}+u_{11}\frac{\partial x_1}{\partial p_r}+u_{12}\frac{\partial x_2}{\partial p_r}+\dots+u_{1n}\frac{\partial x_n}{\partial p_r}&=0\\ \dots\dots\dots\dots\dots\dots&\\ -p_r\frac{\partial\mu}{\partial p_r}+u_{1r}\frac{\partial x_1}{\partial p_r}+u_{2r}\frac{\partial x_2}{\partial p_r}+\dots+u_{rn}\frac{\partial x_n}{\partial p_r}&=\mu\\ \dots\dots\dots\dots\dots\dots&\\ -p_n\frac{\partial\mu}{\partial p_r}+u_{1n}\frac{\partial x_1}{\partial p_r}+u_{2n}\frac{\partial x_2}{\partial p_r}+\dots+u_{nn}\frac{\partial x_n}{\partial p_r}&=0 \end{aligned}\right\}. \tag{7.1}$$

Solving and simplifying as before,

$$\frac{\partial x_s}{\partial p_r}=\frac{1}{U}(-x_r\mu U_s+\mu U_{rs}) \quad (r \text{ and } s=1,2,3,\dots,n).$$

Applying (6.3), this can be written

$$\frac{\partial x_s}{\partial p_r}=-x_r\frac{\partial x_s}{\partial M}+\mu\frac{U_{rs}}{U} \quad (r \text{ and } s=1,2,3,\dots,n). \tag{7.2}$$

This equation, originally due to Slutsky, may be regarded as the Fundamental Equation of Value Theory. It gives us the effect of a change in the price of a commodity x_r on the individual's demand for another commodity x_s, split up into two terms, which we have called the Income Effect and the Substitution Effect respectively. Since $x_r = dM/dp_r$, when M is not taken as given, but all x's and all other p's are taken as given, it follows from the equation that the substitution term represents the effect on the demand for x_s of a change in the price of x_r combined with such a change in income as would enable the consumer, if he chose, to buy the same quantities of all goods as before, in spite of the change in p_r. It is obvious that this change in income will be smaller, the less important is x_r in the consumer's budget.

By putting r and s equal (there is no reason why we should not do so), the same equation can be used to split up the effect of a change in the price of x_r on the demand for x_r itself. The equation will then read

$$\frac{\partial x_r}{\partial p_r}=-x_r\frac{\partial x_r}{\partial M}+\frac{\mu U_{rr}}{U}.$$

It follows directly from the stability conditions that the substitution term in this equation must be negative.

8. *Properties of the substitution term.* Most of the rest of the theory of consumer's demand consists in working out the properties of this fundamental equation. First of all, it will be convenient to write it in an alternative form. The substitution term $\mu U_{rs}/U$ is in fact invariant against a substitution of $\phi(u)$ for u as the utility function; consequently it is better to write it in a form which does not make direct reference to a particular utility function. I shall therefore write it in the non-committal form x_{rs}; so that the equations become

$$\left.\begin{aligned} \frac{\partial x_s}{\partial p_r}&=-x_r\frac{\partial x_s}{\partial M}+\mathrm{x}_{rs},\\ \frac{\partial x_r}{\partial p_r}&=-x_r\frac{\partial x_r}{\partial M}+\mathrm{x}_{rr}. \end{aligned}\right\} \tag{8.1}$$

This is the form in which we shall find it most convenient to use them in our further work.[1]

[1] From some points of view, but not (I think) from all, there is an advantage

Two properties of the substitution term follow at once from what has been said already. I shall first of all write down these properties and then go on to work out some others.

(1) Since the determinants U_{rs} and U are both symmetrical between r and s, X_{rs} is also symmetrical; that is to say, $X_{sr} = X_{rs}$. The substitution terms in $\partial x_s/\partial p_r$ and $\partial x_r/\partial p_s$ are therefore identical; but the income terms are not, in general, equal. Thus, in order for $\partial x_s/\partial p_r$ and $\partial x_r/\partial p_s$ to be equal, it is necessary that $x_r(\partial x_s/\partial M)$ and $x_s(\partial x_r/\partial M)$ should be equal. This implies that $(M/x_s)\partial x_s/\partial M$ and $(M/x_r)\partial x_r/\partial M$ must be equal; i.e. the elasticities of demand for x_r and x_s with respect to income must be the same.

(2) Since U_{rr}/U is negative, and μ is positive, $X_{rr} < 0$.

(3) The expression

$$0 . U_r + u_1 U_{1r} + u_2 U_{2r} + \ldots + u_n U_{nr}$$

forms a determinant in which two rows are identical; therefore it vanishes. But since $u_s U_{rs} = p_s \mu U_{rs} = p_s U X_{rs}$, we can deduce from this relation a relation between the X's, viz. $\sum_{s=1}^{s=n} p_s X_{rs} = 0$.

$\therefore \sum p_s X_{rs}$ (for all values of s except r) $= -p_r X_{rr}$ which is necessarily positive.

(4) All our work so far has been based upon two only out of the set of stability conditions (4.1), which two conditions we have reduced to one—that U_{rr}/U is negative. How do the other stability conditions come into the picture? Let us proceed to see.

Let $U_{11,22}$ be the co-factor of u_{22} in U_{11}; $U_{11,22,33}$ be the co-factor of u_{33} in $U_{11,22}$; and so on. Then the stability conditions tell us that

$$\frac{U_{11}}{U}, \quad \frac{U_{11,22}}{U}, \quad \frac{U_{11,22,33}}{U}, \ldots$$

are alternatively negative and positive.

It follows (by a well-known property of reciprocal determinants)[1] that

$$\frac{U_{11}}{U}, \quad \frac{1}{U^2}\begin{vmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{vmatrix}, \quad \frac{1}{U^3}\begin{vmatrix} U_{11} & U_{12} & U_{13} \\ U_{12} & U_{22} & U_{23} \\ U_{13} & U_{23} & U_{33} \end{vmatrix}, \ldots$$

are alternatively negative and positive.

to be gained if we express the fundamental equation in elasticity form, as can easily be done by multiplying the equation through by p_r/x_s, and grouping the resulting expression into fractions which are independent of units. In my French pamphlet, *La Théorie mathématique de la Valeur* (Hermann, 1937), I have set out a large part of the following argument, using the elasticity method of statement. So the reader can take his choice.

[1] Cf., for example, Burnside and Panton, vol. ii, p. 42.

But these are the conditions that a quadratic form such as

$$\sum_{r=1}^{r=m} \sum_{s=1}^{s=m} x_r x_s \frac{U_{rs}}{U}$$

should be necessarily negative, for all values of the x's. (Cf. 2(2) above.)

Consequently $\sum_1^m \sum_1^m \lambda_r \lambda_s X_{rs} < 0$, for all values of the arbitrary coefficients λ, and for all values of m up to and including n.

We have thus accumulated four rules which must be obeyed by the substitution terms:

(1) $X_{sr} = X_{rs}$; (2) $X_{rr} < 0$; (3) $\sum_{s=1}^{s=n} p_s X_{rs} = 0$;

(4) $\sum_1^m \sum_1^m \lambda_r \lambda_s X_{rs} < 0$ for all values of m up to n.

It will be observed that rule (2) is a special case of rule (4). Among other values, the λ's may take values equal to the p's. Thus we have as a special case of rule (4)

$$\sum_1^m \sum_1^m p_r p_s X_{rs} < 0 \text{ for all values of } m \text{ less than } n.$$

It follows from this, together with the third rule, that

$$\sum_{r=1}^{r=m} \sum_{s=m+1}^{s=n} p_r p_s X_{rs} > 0.$$

This last inequality may be expressed in words in the following way. If we divide the n commodities into two groups in any possible manner and form the expression $p_r p_s X_{rs}$ (x_r being taken from one group and x_s from the other), then $\sum \sum p_r p_s X_{rs}$ (where r and s vary in every possible way within their respective groups) must be positive.

If we consider a change in prices which is such that the changes in different prices *compensate*, leaving the consumer on the same indifference level after the change as before, the income-term in the fundamental equation vanishes, and we have

$$dx_r = \sum_s \frac{\partial x_r}{\partial p_s} dp_s = \sum_s X_{rs} dp_s.$$

Thus $$\sum_r dx_r dp_r = \sum_r \sum_s X_{rs} dp_r dp_s,$$

which by rule (4) is necessarily negative. This is the same proposition as we reached by another route on p. 52 above.

9. *Complementarity*. As in the text of this book, I say that two goods x_r and x_s are substitutes from the point of view of a particular consumer if his $X_{rs} > 0$; complementary if his $X_{rs} < 0$. It follows at once from Rule (5) that, while it is possible for all other goods consumed to be substitutes for x_r, it is not possible for them all to be complementary with it. And it follows from Rule (6) that there is a further limit on the amount of complementarity possible. There are a large number of ways in which the substitution terms between pairs of goods can be taken together in groups, within which the pairs that are substitutes must outweigh the pairs that can be complements. There are $\frac{1}{2}n(n-1)$ different pairs of goods which can be selected out of a group of n goods; these $\frac{1}{2}n(n-1)$ pairs can be taken together in groups of this sort in

$$\tfrac{1}{2}(C_1^n+C_2^n+\ldots+C_{n-2}^n+C_{n-1}^n) = 2^{n-1}-1$$

different ways. The $\frac{1}{2}n(n-1)$ expressions $p_r p_s X_{rs}$ $(r \neq s)$ need not all be positive; but there are $2^{n-1}-1$ different collections of them whose sums must be positive. This is the sense in which substitution is dominant throughout the system as a whole.

10. *The demand for a group of goods.* We have still to consider the most important application of Rule (4). To begin with, it follows from our fundamental equation that the *value* of the increment in the demand for x_s which results from a given *proportionate* change in the price of x_r

$$= p_r p_s \frac{\partial x_s}{\partial p_r} = -p_r x_r \cdot p_s \frac{\partial x_s}{\partial M} + p_r p_s X_{rs} \qquad (10.1)$$

Here $p_r x_r$ is the amount spent on x_r; $p_s(\partial x_s/\partial M)$ measures the increment in the amount spent on x_s which would result from a rise in income.

Now suppose that the prices of a group of goods $x_1, x_2, \ldots, x_m$ $(m < n)$ rise, all in the same proportion. Then the value of the increment in the demand for one of these goods x_s $(s < m)$ is given by summing the above expressions:

$$\sum_{r=1}^{r=m} p_r p_s \frac{\partial x_s}{\partial p_r} = -\left(\sum_1^m p_r x_r\right) p_s \frac{\partial x_s}{\partial M} + \sum_{r=1}^{r=m} p_r p_s X_{rs}.$$

The value of the increment in demand for the whole group taken together is given by summing again:

$$\sum_{s=1}^{s=m} \sum_{r=1}^{r=m} p_r p_s \frac{\partial x_s}{\partial p_r} = -\left(\sum_1^m p_r x_r\right)\left(\sum_1^m p_s \frac{\partial x_s}{\partial M}\right) + \sum_{r=1}^{r=m} \sum_{s=1}^{s=m} p_r p_s X_{rs}. \qquad (10.2)$$

This has identically the same form as (10.1) and has a corresponding interpretation. Further, since the r's and s's are summed over the same group of goods, it follows from Rule (4) that the substitution term in (10.2) is necessarily negative.

Thus we have demonstrated mathematically the very important principle, used extensively in the text, that if the prices of a group of goods change in the same proportion, that group of goods behaves just as if it were a single commodity.

11. *The supply side.* Suppose now that an individual, instead of coming to the market with a given quantity of money, which does not vary when prices vary, comes with a certain quantity of goods for sale, so that the amount he has available for expenditure is affected by market prices. To take the general case, suppose that he starts off with quantities $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n$ of the n goods. As a result of trading, he will increase or diminish these quantities, so as to acquire a preferred collection $x_1, x_2, x_3, \ldots, x_n$ as before. The first of the equilibrium equations (6.1) must then read

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = p_1 \bar{x}_1 + p_2 \bar{x}_2 + \ldots + p_n \bar{x}_n. \qquad (11.1)$$

That is the only alteration which has to be made to the system.

This alteration amounts to replacing M by the quantity $\sum p_r \bar{x}_r$, which is no longer independent of prices. Therefore, when we differentiate the equations, we can no longer put $\partial M/\partial p_r = 0$, but must write $\partial M/\partial p_r = \bar{x}_r$. The first equation of (7.1) then becomes

$$p_1 \frac{\partial x_1}{\partial p_r} + p_2 \frac{\partial x_2}{\partial p_r} + \ldots + p_n \frac{\partial x_n}{\partial p_r} = \bar{x}_r - x_r.$$

And instead of the equation (8.1) we shall get

$$\frac{\partial x_s}{\partial p_r} = (\bar{x}_r - x_r)\frac{\partial x_s}{\partial M} + X_{rs}.$$

This only differs from our first fundamental equation in that the income term is now weighted by the *net* amount of x_r acquired.

12. *Market demand.* It is one of the most obvious conveniences of our Fundamental Equation that it can be applied directly to deal with the effect of a change in price on the demand from a group of individuals. If the summations are taken over all members of the group,

$$\frac{\partial}{\partial p_r}\left(\sum x_s\right) = \sum \frac{\partial x_s}{\partial p_r} = \sum \left[(\bar{x}_r - x_r)\frac{\partial x_s}{\partial M}\right] + \sum X_{rs}. \qquad (12.1)$$

The income term corresponds to the effect on the demand of the group for x_s, when the group's income is increased, but the increment in income is divided among its members in proportion to each individual's previous net demand for x_r. The substitution term is a mere aggregate of individual substitution terms; it must therefore obey the same rules as its components do. If we write the group substitution term $\sum \mathbf{X}_{rs}$ in the form $\mathbf{X}_{rs}$, we shall have exactly corresponding rules

(1) $\mathbf{X}_{sr} = \mathbf{X}_{rs}$, (2) $\mathbf{X}_{rr} < 0$,

(3) $\sum_{s=1}^{s=n} p_s \mathbf{X}_{rs} = 0$, (4) $\sum_1^m \sum_1^m p_r p_s \mathbf{X}_{rs} < 0$,

(5) $\sum_{s \neq r} p_s \mathbf{X}_{rs} > 0$, (6) $\sum_{r=1}^{r=m} \sum_{s=m+1}^{s=n} p_r p_s \mathbf{X}_{rs} > 0$.

Appendix to Chapter IV

13. *Equilibrium of exchange.* Here it is only necessary for us to restate the classical argument of Walras in our own terms.

We have N individuals bringing to the market various quantities of n goods, and exchanging them under conditions of perfect competition. Since we are writing the quantity of the rth good originally at the disposal of a representative individual $\bar{x}_r$, and the amount he ultimately retains x_r ($x_r > \bar{x}_r$ if he is a buyer of that good, $< \bar{x}_r$ if he is a seller), let us write the total quantity originally brought by all individuals together $\bar{X}_r$, the total amount ultimately retained X_r.

The prices of the n goods we shall denote as before by $p_1, p_2, \ldots, p_n$. But it must be remembered that one good (say x_n) has to be taken as standard of value. Therefore $p_n = 1$. The remaining prices $p_1, p_2, \ldots, p_{n-1}$ have to be determined.

If the system is to be in equilibrium, the demand for every commodity must equal the supply.

$$\therefore X_r = \bar{X}_r \quad (r = 1, 2, 3, \ldots, n). \tag{13.1}$$

This gives us n equations corresponding to the n goods; but there are only $n-1$ prices to be determined. However, one equation follows from the rest. Among the equations of equilibrium of a representative individual is the equation

$$\sum_1^n p_r x_r = \sum_1^n p_r \bar{x}_r. \tag{11.1}$$

Summing these equations over all individuals, we have

$$\sum_1^n p_r X_r = \sum_1^n p_r \bar{X}_r.$$

Since this last equation must necessarily hold, whether the equations (13.1) are satisfied or not, it follows that if $n-1$ of the equations (13.1) are satisfied, the nth equation must be satisfied too. There are therefore only $n-1$ equations to determine the $n-1$ prices.

Appendix to Chapter V

14. *The stability of exchange equilibrium.* Since $\bar{X}_r$ can be taken as constant, the conditions for the stability of exchange can be got by examining the sign of dX_r/dp_r. In order for equilibrium to be perfectly stable, dX_r/dp_r must be negative

(1) when all other prices are unchanged;
(2) when p_s is adjusted so as to maintain equilibrium in the market for x_s, but all other prices are unchanged;
(3) when p_s and p_t are similarly adjusted;

and so on, until we have adjusted all prices, excepting p_r (and of course p_n, which is necessarily 1).

The third of these conditions, for example, implies that dX_r/dp_r is negative, when

$$\left.\begin{aligned} \frac{dX_r}{dp_r} &= \frac{\partial X_r}{\partial p_r} + \frac{\partial X_r}{\partial p_s}\frac{dp_s}{dp_r} + \frac{\partial X_r}{\partial p_t}\frac{dp_t}{dp_r} \\ 0 &= \frac{\partial X_s}{\partial p_r} + \frac{\partial X_s}{\partial p_s}\frac{dp_s}{dp_r} + \frac{\partial X_s}{\partial p_t}\frac{dp_t}{dp_r} \\ 0 &= \frac{\partial X_t}{\partial p_r} + \frac{\partial X_t}{\partial p_s}\frac{dp_s}{dp_r} + \frac{\partial X_t}{\partial p_t}\frac{dp_t}{dp_r} \end{aligned}\right\}. \tag{14.1}$$

Eliminating dp_s/dp_r and dp_t/dp_r

$$\frac{dX_r}{dp_r} = \begin{vmatrix} \frac{\partial X_r}{\partial p_r} & \frac{\partial X_r}{\partial p_s} & \frac{\partial X_r}{\partial p_t} \\ \frac{\partial X_s}{\partial p_r} & \frac{\partial X_s}{\partial p_s} & \frac{\partial X_s}{\partial p_t} \\ \frac{\partial X_t}{\partial p_r} & \frac{\partial X_t}{\partial p_s} & \frac{\partial X_t}{\partial p_t} \end{vmatrix} : \begin{vmatrix} \frac{\partial X_s}{\partial p_s} & \frac{\partial X_s}{\partial p_t} \\ \frac{\partial X_t}{\partial p_s} & \frac{\partial X_t}{\partial p_t} \end{vmatrix}.$$

This gives us one of the expressions which must be negative, in order for the system to be stable.

Taking all similar conditions together, and remembering that they must hold for the market in every x_r $(r = 1, 2, 3, \ldots, n-1)$, the stability conditions emerge in a more convenient form. It is necessary for the Jacobian determinants

$$\frac{\partial X_r}{\partial p_r}, \quad \begin{vmatrix} \frac{\partial X_r}{\partial p_r} & \frac{\partial X_r}{\partial p_s} \\ \frac{\partial X_s}{\partial p_r} & \frac{\partial X_s}{\partial p_s} \end{vmatrix}, \quad \begin{vmatrix} \frac{\partial X_r}{\partial p_r} & \frac{\partial X_r}{\partial p_s} & \frac{\partial X_r}{\partial p_t} \\ \frac{\partial X_s}{\partial p_r} & \frac{\partial X_s}{\partial p_s} & \frac{\partial X_s}{\partial p_t} \\ \frac{\partial X_t}{\partial p_r} & \frac{\partial X_t}{\partial p_s} & \frac{\partial X_t}{\partial p_t} \end{vmatrix}, \ldots \tag{14.2}$$